CLOSED FORM SOLUTIONS FOR DYNAMIC ANALYSIS OF SIMPLY SUPPORTED MINDLIN PLATES

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1. INTRODUCTION

The object of this first chapter is to describe in detail some closed form solutions for dynamic analysis of simply supported rectangular Mindlin plates which rest on elastic Winkler foundations. These plates have specially orthotropic form and four different types of loading are considered, using as forcing functions: a sine pulse, a step pulse, a triangular pulse, a stepped triangular pulse or an exponential pulse. Other types of loading and forcing functions can easily by incorporated. The solutions given in this chapter include a free vibration and an initially stressed vibration analysis, as well as a buckling analysis. In Section 4 of this chapter a program called TRAM is presented for the closed form solution. The solutions obtained from this program may be used to check the accuracy of the finite strip and finite element solutions presented in Chapters 2 and 3 of this book. Some examples illustrating the use of program TRAM are given in Section 5.

2. GOVERNING EQUATIONS

Mindlin plate theory [1] allows for transverse shear deformation and thus offers an alternative to classical Kirchhoff thin plate theory [2]. The main assumptions are that:

(a) displacements are small compared with the plate thickness,

(b) the stress normal to the midplane of the plate is negligible, and

(c) normals to the midplane before deformation remain straight but not necessarily normal to the midplane after deformation.

The third assumption leads to the following displacement field,

$$\bar{u}(x,y,z) = z\theta_x(x,y) \tag{2.1a}$$

$$\bar{v}(x,y,z) = z\theta_y(x,y) \tag{2.1b}$$

$$\bar{w}(x,y,z) = w(x,y) \tag{2.1c}$$

in which x and y are the rectangular coordinates in the plane of the plate, z is the thickness-direction coordinate measured downwards from the midplane; $\bar{u}$, $\bar{v}$ and $\bar{w}$ are the displacements in the x, y and z directions respectively, w is the corresponding midplane displacement, and θ_x and θ_y are the normal rotations in the xz- and yz-planes respectively due to bending.

By making use of the assumption that $\sigma_z = 0$, the constitutive equations at a typical point (x, y, z) in a Mindlin plate may be expressed as

$$\boldsymbol{\sigma} = \mathbf{Q}\,\boldsymbol{\epsilon} \tag{2.2a}$$

where the stress vector $\boldsymbol{\sigma}$ has the form

$$\boldsymbol{\sigma} = [\sigma_x, \sigma_y, \tau_{xy}]^T \tag{2.2b}$$

the matrix of reduced in-plane stiffnesses for plane stress (i.e. assuming that $\sigma_z = 0$) may be written as

$$\mathbf{Q} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \tag{2.2c}$$

the strain vector $\boldsymbol{\epsilon}$ is expressed as

$$\boldsymbol{\epsilon} = [\bar{u}_{,x}, \bar{v}_{,y}, \bar{u}_{,y} + \bar{v}_{,x}]^T \tag{2.2d}$$

Using relations (2.1), (2.2d) may be rewritten as

$$\boldsymbol{\epsilon} = z\,\boldsymbol{\epsilon}_f \tag{2.2e}$$

where

$$\boldsymbol{\epsilon}_f = [\theta_{x,x}, \theta_{y,y}, (\theta_{x,y} + \theta_{y,x})]^T \tag{2.2f}$$

Note that $\theta_{x,x} = \partial\theta_x/\partial x$ etc. The transverse shear stress-strain relations have the form

$$\boldsymbol{\tau} = \mathbf{C}\boldsymbol{\gamma} \tag{2.3a}$$

where the shear stress vector $\boldsymbol{\tau}$ is written as

$$\boldsymbol{\tau} = [\tau_{yz}, \tau_{xz}]^T \tag{2.3b}$$

The matrix of elastic constants $\mathbf{C}$ can be written as

$$\mathbf{C} = \begin{bmatrix} C_{44} & C_{45} \\ C_{45} & C_{55} \end{bmatrix} \tag{2.3c}$$

The vector of shear strains has the form

$$\boldsymbol{\gamma} = (\bar{v}_{,z} + \bar{w}_{,y}, \bar{u}_{,z} + \bar{w}_{,x})^T \tag{2.3d}$$

Again, using expressions (2.1), the shear strain vector can be rewritten as

$$\boldsymbol{\gamma} = \boldsymbol{\epsilon}_s \tag{2.3e}$$

or

$$\boldsymbol{\epsilon}_s = [\theta_y + w_{,y}, \theta_x + w_{,x}]^T \tag{2.3f}$$

For a Mindlin plate of thickness h and area A, the strain energy can be written as

$$\begin{aligned} S.E. &= 1/2 \int_A \Big(\int_{-h/2}^{h/2} \boldsymbol{\sigma}^T \boldsymbol{\epsilon}\, dz\Big) dA + 1/2 \int_A \Big(\int_{-h/2}^{h/2} \boldsymbol{\gamma}^T \boldsymbol{\tau}\, dz\Big) dA \\ &= 1/2 \int_A \Big(\int_{-h/2}^{h/2} \boldsymbol{\epsilon}_f^T\, (z^2\mathbf{Q})\, \boldsymbol{\epsilon}_f\, dz\Big) dA + 1/2 \int_A \Big(\int_{-h/2}^{h/2} \boldsymbol{\epsilon}_s^T\, \mathbf{C}\, \boldsymbol{\epsilon}_s\, dz\Big) dA \\ &= 1/2 \int_A \boldsymbol{\epsilon}_f^T\, \mathbf{D}_f\, \boldsymbol{\epsilon}_f\, dA + 1/2 \int_A \boldsymbol{\epsilon}_s^T\, \mathbf{D}_s\, \boldsymbol{\epsilon}_s\, dA \ cr \end{aligned} \tag{2.4}$$

The plate constitutive equations can be written as

$$\boldsymbol{\sigma}_f = \mathbf{D}_f\,\boldsymbol{\epsilon}_f \tag{2.5a}$$

where the bending moments are written as

$$\boldsymbol{\sigma}_f = [M_x, M_y, M_{xy}]^T \tag{2.5b}$$

in which

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} z(\sigma_x, \sigma_y, \tau_{xy})\, dz \tag{2.5c}$$

The matrix of flexural rigidities has the form

$$\mathbf{D}_f = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \tag{2.5d}$$

in which

$$D_{ij} = \int_{-h/2}^{h/2} Q_{ij}\, z^2\, dz \qquad (i,j = 1,2,6) \tag{2.5e}$$

Further, it is possible to write

$$\boldsymbol{\sigma}_s = \mathbf{D}_s\,\boldsymbol{\epsilon}_s \tag{2.6a}$$

where the shear forces may be written as

$$\boldsymbol{\sigma}_s = [Q_y, Q_x]^T \tag{2.6b}$$

and

$$(Q_y, Q_z) = \int_{-h/2}^{h/2} (\tau_{yz}, \tau_{xz})\, dz \tag{2.6c}$$

The matrix of shear rigidities can be written as

$$\mathbf{D}_s = \begin{bmatrix} S_{44} & S_{45} \\ S_{45} & S_{55} \end{bmatrix} \tag{2.6d}$$

where

$$S_{ij} = \int_{-h/2}^{h/2} C_{ij}\, dz \qquad (i,j = 4,5) \tag{2.6e}$$

Note that for specially orthotropic plates D_{16}, D_{61}, D_{26} and $D_{62} = 0$. For homogeneous plates D_{ij} and S_{ij} may be evaluated from (2.5e) and (2.6e), and for sandwich, laminated and voided plates one can use formulae which are given in Reference 3.

Note also that the strain resultants given in (2.2f) and (2.3f) are the direct curvatures $\theta_{x,x}$ and $\theta_{y,y}$ in the x and y directions respectively, the twisting curvature $(\theta_{y,x} + \theta_{x,y})$ and the shear rotations $(\theta_y + w_{,y})$ and $(\theta_x + w_{,x})$ in the yz- and xz–planes respectively.

The equations of motion for a Mindlin plate resting on an elastic Winkler foundation of modulus K, subjected due to the uniform initial stress resultants F_x and F_y may be written as

$$Q_{x,x} + Q_{y,y} + F_x w_{,xx} + F_y w_{,yy} + Kw + q = P\ddot{w} \tag{2.7a}$$

$$M_{x,x} + M_{xy,y} - Q_x + m_x = I\ddot{\theta}_x \tag{2.7b}$$

$$M_{xy,x} + M_{y,y} - Q_y + m_y = I\ddot{\theta}_y \tag{2.7c}$$

in which q, m_x and m_y are the distributed lateral loads and couples. In (2.7), differentiation with respect to x or y is denoted by a comma while differentiation with respect to time is denoted by a superposed dot, and inertias are given by

$$(P, I) = \int_{-h/2}^{h/2} \rho(1, z^2) dz \tag{2.8}$$

The closed form solution given later in the chapter is concerned with simply supported rectangular plates of uniform thickness with dimensions a and b for which the boundary conditions are given as

$$w = \theta_{x,x} = 0 \qquad x = 0, a \tag{2.9a}$$

$$w = \theta_{y,y} = 0 \qquad y = 0, b \tag{2.9b}$$

By combining (2.5a), (2.6a) and (2.7) the governing equations may be expressed as

$$S_{55}\theta_{x,x} + (S_{55} + F_x)w_{,xx} + S_{44}\theta_{y,y} + (S_{44} + F_y)w_{,yy} + Kw + q = P\ddot{w} \tag{2.10a}$$

$$D_{11}\theta_{x,xx} + D_{66}\theta_{x,yy} + (D_{12} + D_{66})\theta_{y,xy} - S_{55}\theta_x - S_{55}w_{,x} + m_x = I\ddot{\theta}_x \tag{2.10b}$$

$$(D_{12} + D_{66})\theta_{x,xy} + D_{66}\theta_{y,xx} + D_{22}\theta_{y,yy} - S_{44}\bar{\theta}_y - S_{44}w_{,y} + m_y = I\ddot{\theta}_y \tag{2.10c}$$

For the closed form solution functions w, θ_x and θ_y are sought which satisfy (2.9) and (2.10).

3. CLOSED FORM SOLUTION

3.1 Vibration Analysis

Here the closed form solution given by Dobyns [4] is presented for a simply supported rectangular plate of uniform thickness, which has dimensions a and b, and which rests on an elastic Winkler foundation. Solutions to equations (2.10) that satisfy boundary conditions (2.9) are given as

$$\theta_x = \Phi_x^{mn} e^{i\omega_{mn} t} \tag{3.1a}$$

$$\theta_y = \Phi_y^{mn} e^{i\omega_{mn} t} \tag{3.1b}$$

$$w = W^{mn} e^{i\omega_{mn} t} \tag{3.1c}$$

where

$$\Phi_x^{mn} = A_{mn} \cos(m\pi x/a) \sin(n\pi y/b) \tag{3.2a}$$

$$\Phi_y^{mn} = B_{mn} \sin(m\pi x/a) \cos(n\pi y/b) \tag{3.2b}$$

$$W^{mn} = C_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \tag{3.2c}$$

Substitution of the assumed displacements (3.1) into the equations of motion (2.10) results in

$$\begin{aligned} &D_{11}\Phi^{mn}_{x,xx} + D_{66}\Phi^{mn}_{x,yy} + (D_{12} + D_{66})\Phi^{mn}_{y,xy} - S_{55}\Phi^{mn}_x - S_{55}W^{mn}_{,x} = -\omega^2_{mn} I\Phi^{mn}_x \\ &(D_{12} + D_{66})\Phi^{mn}_{x,xy} + D_{66}\Phi^{mn}_{y,xx} + D_{22}\Phi^{mn}_{y,yy} - S_{44}\Phi^{mn}_y - S_{44}W^{mn}_{,y} = -\omega^2_{mn} I\Phi^{mn}_y \\ &S_{55}\Phi^{mn}_{x,x} + (S_{55} + F_x)W^{mn}_{,xx} + S_{44}\Phi^{mn}_{y,y} + (S_{44} + F_y)W^{mn}_{,yy} + KW^{mn} = -\omega^2_{mn} PW^{mn} \end{aligned} \tag{3.3}$$

Upon substitution of (3.2) into the equilibrium equations of (3.3) it is possible to obtain a set of homogeneous equations that may be solved for the natural frequencies of vibration.

$$\begin{bmatrix} (P_{11} - \omega^2_{mn} I) & P_{12} & P_{13} \\ P_{12} & (P_{22} - \omega^2_{mn} I) & P_{23} \\ P_{13} & P_{23} & (P_{33} - \omega^2_{mn} P) \end{bmatrix} \begin{bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.4}$$

where

$$P_{11} = D_{11}(m\pi/a)^2 + D_{66}(n\pi/b)^2 + S_{55}$$
$$P_{12} = (D_{12} + D_{66})(m\pi/a)(n\pi/b)$$
$$P_{13} = S_{55}(m\pi/a)$$
$$P_{22} = D_{66}(m\pi/a)^2 + D_{22}(n\pi/b)^2 + S_{44}$$
$$P_{23} = S_{44}(n\pi/b)$$
$$P_{33} = (S_{55} + F_x)(m\pi/a)^2 + (S_{44} + F_y)(n\pi/b)^2 + K \quad (3.5)$$

Three eigenvalues and their respective eigenvectors result from (3.4) for each m, n pair.

If the rotatory interia (I) is neglected (Mindlin [1] has shown that rotatory inertia has little effect in isotropic plates and it is popularly assumed that the same holds true for orthotropic plates), only one eigenvalue and its eigenvector results for each m, n. The frequency of vibration is given as

$$\omega_{mn}^2 = (QP_{33} + 2P_{12}P_{23}P_{13} - P_{22}P_{13}^2 - P_{11}P_{23}^2)/(PQ) \quad (3.6)$$

where $Q = (P_{11}P_{22} - P_{12}^2)$ and the eigenvectors associated with the natural frequency are

$$A_{mn} = C_{mn}(P_{12}P_{23} - P_{22}P_{13})/(P_{11}P_{22} - P_{12}^2) \quad (3.7a)$$

$$B_{mn} = C_{mn}(P_{12}P_{13} - P_{11}P_{23})/(P_{11}P_{22} - P_{12}^2) \quad (3.7b)$$

when normalised to C_{mn}.

The orthogonality condition for the principal modes is given by

$$(\omega_{mn}^2 - \omega_{pq}^2)\int_0^a\int_0^b (PW^{mn}W^{pq} + I\Phi_x^{mn}\Phi_x^{pq} + I\Phi_y^{mn}\Phi_y^{pq})dxdy = 0 \quad (3.9)$$

so that if $m, n \neq p, q$ then the integral is zero.

3.2 Stability Analysis

If a plate is subjected to uniform stress resultants F_x and F_y the frequency of vibration will be either increased or decreased, depending on the direction of the stress resultants. From (3.5) and (3.6) it is obvious that for the compression the frequency of vibration will be reduced. Thus, when the frequency is zero, either F_x or F_y, or a combination of both will be the critical buckling load. By putting ω_{mn} equal to zero in (3.6) the equation can be reduced to the form

$$F_x(m\pi/a)^2 + F_y(n\pi/b)^2 = R + S \quad (3.9a)$$

where

$$R = (2P_{12}P_{23}P_{13} - P_{22}P_{13}^2 - P_{11}P_{23}^2)/(P_{11}P_{22} - P_{12}^2) \quad (3.9b)$$

and

$$S = (S_{55}(m\pi/a)^2 + S_{44}(n\pi/b)^2 + K) \quad (3.9c)$$

Assuming that there are given ratios between the uniform stress resultants F_x and F_y and the critical buckling load F so that

$$F_x = \alpha F \quad (3.10)$$

$$F_y = \beta F \quad (3.11)$$

the critical buckling load is given by

$$F_{crit} = (R + S)/[\alpha(m\pi/a)^2 + \beta(n\pi/b)^2] \quad (3.12)$$

3.3 Dynamic Transient Analysis

The solution to the equations of motion can be sought as a product of two functions, a function of position and a function of time, as follows:

$$\theta_x(x, y, t) = \sum_m\sum_n \Phi_x^{mn}(x, y)T_{mn}(t) \quad (3.13a)$$

$$\theta_y(x, y, t) = \sum_m\sum_n \Phi_y^{mn}(x, y)T_{mn}(t) \quad (3.13b)$$

$$w(x, y, t) = \sum_m\sum_n W^{mn}(x, y)T_{mn}(t) \quad (3.13b)$$

where $T_{mn}(t)$ is a time dependent generalised coordination and $\sum_m$ implies a summation from $m = 1$ to ∞. Substituting equations (3.13) into the equilibrium equations (2.10) and using (3.3) gives

$$-\sum_m\sum_n \omega_{mn}^2\Phi_x^{mn}T_{mn} + m_x/I = \sum_m\sum_n \Phi_x^{mn}\ddot{T}_{mn} \quad (3.14a)$$

$$-\sum_m\sum_n \omega_{mn}^2\Phi_y^{mn}T_{mn} + m_y/I = \sum_m\sum_n \Phi_y^{mn}\ddot{T}_{mn} \quad (3.14b)$$

$$-\sum_m\sum_n \omega_{mn}^2 W^{mn}T_{mn} + q/P = \sum_m\sum_n W^{mn}\ddot{T}_{mn} \quad (3.14c)$$

The distributed loads m_x, m_y and q are expanded in terms of the generalised forces $Q_{mn}(t)$ as

$$m_x/I = \sum_m\sum_n Q_{mn}(t)\Phi_x^{mn}(x, y) \quad (3.15a)$$

$$m_y/I = \sum_m\sum_n Q_{mn}(t)\Phi_y^{mn}(x, y) \quad (3.15b)$$

$$q/P = \sum_m\sum_n Q_{mn}(t)W^{mn}(x, y) \quad (3.15c)$$